Approximation of functions:
Problem Sheet 1

1. The following exercises from LNT's book: 1.1*, 1.2*, 2.1*, 3.1, 3.5*, 3.6, 3.7, 3.10

   Exercises with a * require chebfun computing.

2. 2009 Finals

   Let $u \in C^{(n+1)}[-1,1]$ and $x_0, \ldots, x_n$ be distinct, ordered points in $[-1,1]$ with $u_r = u(x_r)$. Let P_n be the space of polynomials of degree less than or equal to n. Define functions

$$\pi_n(x) = \prod_{s=0}^{n}(x - x_s) \in P_{n+1}, \qquad \pi_{n,r}(x) = \frac{\pi_n(x)}{x - x_r} \in P_n.$$

   (a) Construct the Lagrange interpolating polynomial $p_n(x) \in P_n$ that satisfies

$$p_n(x_r) = u_r, \quad r = 0, 1, \ldots, n.$$

   (b) Show that p_n can be expressing in Barycentric form

$$p_n(x) = \sum_{s=0}^{n} \frac{w_s u_s}{x - x_s} \Big/ \sum_{s=0}^{n} \frac{w_s}{x - x_s},$$

   where the weights w_s are given by

$$w_s^{-1} = \pi_{n,s}(x_s), \quad s = 0, 1, \ldots, n.$$

   (c) Let the interpolation points be uniformly distributed so that $x_r = -1 + rh, \ r = 0, 1, \ldots, n, \quad h = 2/n$. Show that

$$w_s^{-1} = (-1)^s h^n s!(n-s)!, \quad s = 0, 1, \ldots, n.$$

   For large, even n, use Stirlings approximation, $k! \sim \sqrt{2\pi k}(k/e)^k$ to show that

$$\left|\frac{w_0}{w_{n/2}}\right| \sim \sqrt{\pi n} \quad 2^{-n-\frac{1}{2}}.$$

   Comment on the significance of this result for interpolation of u by p_n.

   (d) A different set of interpolation points is given, for even n, by

$$x_r = \cos\theta_r, \quad \theta_r = \frac{r\pi}{n}, \quad r = 0, 1, \ldots, n.$$

   Use induction and properties of the Chebyshev polynomials $T_n(x) = \cos(n\cos^{-1} x)$, (which you should state but not prove) to show that

$$f_n(x) = -2^{1-n}\sin(\cos^{-1} x)\sin(n\cos^{-1} x)$$

is the monic polynomial of degree $n+1$ with roots at $x_0, \ldots, x_n$, and so $\pi_n = f_n$. Show that

$$w_s^{-1} = \frac{\mathrm{d}\pi_n}{\mathrm{d}x}(x_s), \quad s = 0, 1, \ldots, n.$$

Hence use the chain rule to show that the Barycentric weights for interpolation at these points are, after cancellation of common factors,

$$w_s = (-1)^s \delta_s, \quad \text{where} \quad \delta_s = \begin{cases} 1 & s = 1, \ldots, n-1 \\ 1/2 & s = 0, n. \end{cases}$$