## Exchange algorithm

Let $\underline{Z}^k = \{ \xi_0^k \cdots \xi_{m+n+1}^k \}$ be a given reference and $r^k(x)$, $h_k$ be solution of $*$. Choose $\eta$ such that

$$|u(\eta) - r^k(\eta)| > |h_k|$$

and let $\xi_p$ be a point near $\eta$ where error has same sign

Let

$$\underline{Z}^{(k+1)} = \{ \xi_0^k \cdots \xi_{p-1}^k, \eta, \xi_p^k \cdots \xi_{m+n+1}^k \}$$
$$\equiv \{ \xi_0^{k+1} \cdots \xi_{m+n+1}^{k+1} \}$$

and solve $*$ with this reference for $r^{k+1}(x)$, $h_{k+1}$

then $\underline{|h_{k+1}| > |h_k|}$

## Solution of $*$

Rewrite $*$ as

$$\frac{p_m(\xi_r)}{q_n(\xi_r)} + (-1)^r h = u(\xi_r) \qquad r = 0, \ldots, m+n+1$$

or $$p_m(\xi_r) + (-1)^r h\, q_n(\xi_r) = u(\xi_r)\, q_n(\xi_r)$$

Define
$$\underline{X}_r = \begin{bmatrix} 1 \\ \xi_r \\ \vdots \\ \xi_r^m \end{bmatrix}_{m+1 \text{ vector}} \qquad \underline{Y}_s = \begin{bmatrix} 1 \\ \xi_s \\ \vdots \\ \xi_s^n \end{bmatrix}_{n+1 \text{ vector}}$$

$$\underline{p} = \begin{bmatrix} a_0 \\ \vdots \\ a_m \end{bmatrix}_{m+1 \text{ vector}} \qquad \underline{q} = \begin{bmatrix} b_0 \\ \vdots \\ b_n \end{bmatrix}_{n+1 \text{ vector}}$$

$$p_m(\xi_r) = \underline{x}_r^T \underline{p}$$

$$q_n(\xi_r) = y_r^T \underline{q}$$

and $$\underline{x}_r^T \underline{p} + (-1)^r h \, \underline{y}_r^T \underline{q} = u_r \underline{y}_r^T \underline{q}$$

Define matrices

$$S = \begin{bmatrix} x_0^T \\ \vdots \\ x_{m+n+1}^T \end{bmatrix}_{(m+n+2)\times(m+1)}$$

$$P = \begin{bmatrix} u_0 \, y_0^T \\ \vdots \\ u_{m+n+1} \, y_{m+n+1}^T \end{bmatrix}_{(m+n+2)\times(n+1)}$$

$$R = \begin{bmatrix} y_0^T \\ -y_1^T \\ y_2^T \\ \vdots \\ (-1)^{m+n+1} y_{m+n+1}^T \end{bmatrix}_{(m+n+2)\times(n+1)}$$

$$\underbrace{S}_{(m+n+2)\times(m+1)} \cdot \underbrace{\underline{p}}_{(m+1)\text{ vector}} = \underbrace{(P - hR)}_{(m+n+2)\times(n+1)} \underbrace{\underline{q}}_{n+1 \text{ vector}}$$

The matrix $S = \begin{bmatrix} 1 & \xi_0 & \cdots & \xi_0^m \\ \vdots & & & \\ 1 & \xi_{m+n+1} & \cdots & \xi_{m+n+1}^m \end{bmatrix}$ is specified

Define $$v_r = \prod_{\substack{p=0\\p\neq r}}^{m+n+1} \frac{1}{\xi_p - \xi_r} = \frac{1}{\pi'_{m+n+1,r}(\xi_r)}$$

Suppose we interpolate $x^k$ at $\xi_0, \ldots, \xi_{m+n+1}$

$$x^k = \sum_{p=0}^{m+n+1} \xi_p^k \, \ell_{m+n+1,p}(x)$$

or $$x^k = \sum_{p=0}^{m+n+1} \xi_p^k \frac{\pi_{m+n+1,p}(x)}{\pi_{m+n+1,p}(\xi_p)} = \sum_{p=0}^{m+n+1} v_p \xi_p^k \, \pi_{m+n+1,p}(x)$$

but let $\pi_{m+n+1,p} = x^{m+n+1} + h_p \qquad h_p \in P_{m+n}$.

$$\therefore x^k = \left( \sum_{p=0}^{m+n+1} v_p \xi_p^k \right) x^{m+n+1} + \sum_{p=0}^{m+n+1} v_p \xi_p^k h_p$$

If $k = 0, 1, \ldots, m+n$, need coefft of $x^{m+n+1}$ to vanish

$$\therefore \sum_{p=0}^{m+n+1} v_p \xi_p^k = 0 \qquad k = 0, \ldots, m+n.$$

Define $$Q = \begin{bmatrix} v_0 & \cdots & v_{m+n+1} \\ \xi_0 v_0 & \cdots & \xi_{m+n+1} v_{m+n+1} \\ \vdots & & \\ \xi_0^n v_0 & \cdots & \xi_{m+n+1}^n v_{m+n+1} \end{bmatrix}_{(n+1)\times(m+n+2)}$$

$$QS = \begin{bmatrix} v_0 & \cdots & v_{m+n+1} \\ \vdots & & \\ \xi_0^n v_0 & \cdots & \xi_{m+n+1}^n v_{m+n+1} \end{bmatrix} \begin{bmatrix} 1 & \xi_0 & \xi_0^2 & \cdots & \xi_0^m \\ \vdots & & & & \\ 1 & \cdots & & & \xi_{m+n+1}^m \end{bmatrix}$$

$$= \begin{pmatrix} \sum_{p=0}^{m+n+1} v_p & \sum_{p=0}^{m+n+1} \xi_p v_p & \cdots & \sum_{p=0}^{m+n+1} \xi_p^m v_p \\ \vdots & & & \\ \sum_{p=0}^{m+n+1} \xi_p^n v_p & \cdots & & \sum_{p=0}^{m+n+1} \xi_p^{m+n} v_p \end{pmatrix}$$

but all sums are zero

$$\therefore \; QS = \underline{\underline{0}}$$

$\therefore \; Q(P - hR)\underset{\sim}{q} = 0$ eigenvalue problem

$A = PQ$, $B = QR$ just have $(A - hB) = 0$ **

with $$A_{rs} = \sum_{p=0}^{m+n+1} u_p \xi_p^{r+s} v_p$$

$$B_{rs} = \sum_{p=0}^{m+n+1} (-1)^p \xi_p^{r+s} v_p$$

So solve ** for $h$, $\underline{d}$ then

$$S\underset{\sim}{p} = (P - hR)\underset{\sim}{q}$$

$$S^T S\underset{\sim}{p} = S^T(P - hR)\underset{\sim}{q}$$

$$\underset{\sim}{p} = (S^T S)^{-1} S^T (P - hR)\underset{\sim}{q}$$

Choose eigenvalue & eigenvector so $q(x)$ has no singularities

Matlab.

## Lebesgue Constants

For an operator $X[f]$ define the Lebesgue constant

$$\|X\|_\infty = \sup_f \frac{\|X[f]\|_\infty}{\|f\|_\infty}$$

If operator is linear, this is equivalent to

$$\|X\|_\infty = \sup_{\|f\|=1} \|X[f]\|_\infty$$

In polynomial interpolation, define for points $x_0 \ldots x_n$

$$p_n(x) = X[f](x) = \sum_{r=0}^{n}{}' f_r L_{n,r}(x)$$

This operator is linear

$$X[f+g](x) = \sum_{r=0}^{n}{}' (f_r + g_r) L_{n,r}(x)$$

$$= X[f] + X[g] \quad \text{etc}$$

Also

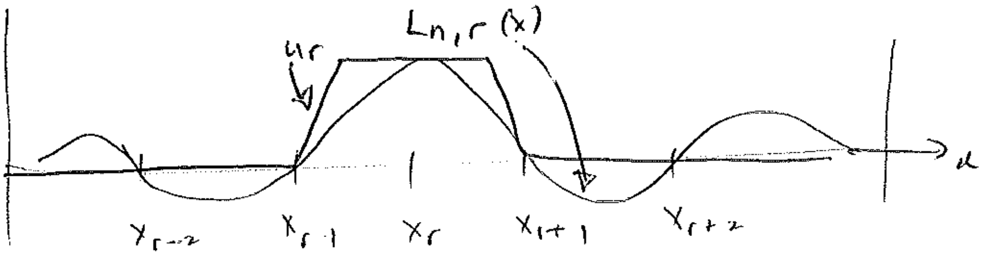
$$\|X\|_\infty = \sup_{\|f\|_\infty = 1} \left| \sum_{r=0}^{n}{}' f_r L_{n,r}(x) \right|$$

$$\leq \max_r |f_r| \max_x \sum_{r=0}^{n}{}' |L_{n,r}(x)|$$

So $\|X\|_\infty \leq \max_x \left| \sum_{r=0}^{n}{}' L_{n,r}(x) \right|$

Now consider function for $\epsilon > 0$

$$u_r(x) = \begin{cases} 0 & x \leq x_{r-1} \\ 1 & x_{r-1} + \epsilon < x \leq x_{r+1} - \epsilon \\ \dfrac{x - x_{r-1}}{\epsilon} & x_{r-1} < x \leq x_{r-1} + \epsilon \\ \dfrac{x_{r+1} - x}{\epsilon} & x_{r+1} - \epsilon < x \leq x_{r+1} \end{cases}$$




$u_r(x)$ is continuous for all $0 < \varepsilon < \min\{x_r - x_{r-1}, x_{r+1} - x_r\}$

and if we interpolate $u_r(x)$ at $x_0, \dots, x_n$ by $q_r(x)$

$$q_r(x) = L_{n,r}(x) \equiv X[u_r]$$

$$\therefore \|X\|_\infty \geq \max_x |L_{n,r}(x)| \qquad r = 0, \dots, n$$

Now let $x$ be fixed and choose $\alpha_0, \dots, \alpha_n$ to be

$$\alpha_r = \operatorname{sign}[L_{n,r}(x)]$$

define $q(x) = \sum' \alpha_r L_{n,r}(x)$

which interpolates $u = \sum' \alpha_r u_r(x)$

but $$|q| = \sum_{r=0}^{n}{}' |L_{n,r}(x)| \leq \|X\|_\infty$$

$$\therefore \max_x \sum_{r=0}^{n}{}' |L_{n,r}(x)| \leq \|X\|_\infty \leq \max_x \sum_{r=0}^{n}{}' |L_{n,r}(x)|$$

i.e. $$\|X\|_\infty = \max_x \sum_{r=0}^{n}{}' |L_{n,r}(x)|$$

Define $$\lambda(x) = \sum_{r=0}^{n}{}' |L_{n,r}(x)|$$

and $\|X\|_\infty = \max_x \lambda(x) = \Lambda_n$ in ATAP.