but at $x_0, \dots x_n$, f must alternate in sign and

is strictly non zero at these points and f is continuous

as it is a polynomial

$\therefore$ f has $n+1$ alternate signed values

$\therefore$ f has n roots, all distinct

but $f \in P_{n-1}$ is a contradiction

$\therefore$ no such v exist

Example! $T_5 = 16x^5 - 20x^3 + 5x$

so $\frac{20x^3 - 5x}{16}$ is the best approximation from P_4 to x^5.

2-12

Barycentric Interpolation (Q 5 LAT)

If we use Lagrange and write the interpolating polynomial as

$$p_n(x) = \sum_{r=0}^{n} f_r \frac{\pi_{n,r}(x)}{\pi_{n,r}(x_r)} \quad \text{and as} \quad \pi_{n,r} = \frac{\pi_n}{x - x_r}$$

$$p_n(x) = \sum_{r=0}^{n} \frac{f_r \, \pi_n(x)}{\pi_{n,r}(x_r)} \cdot \frac{1}{x - x_r}$$

$$= \pi_n(x) \sum_{r=0}^{n} \frac{f_r}{\pi_{n,r}(x_r)(x - x_r)}$$

Let $w_{n,r} = \frac{1}{\pi_{n,r}(x_r)} = \frac{1}{\pi_n'(x_r)}$

so that

$$p_n(x) = \Pi_n(x) \sum_{r=0}^{n} \frac{w_{n,r} f_r}{x - x_r}$$

If we let $f(x) = 1$, we have to also have $p_n(x) = 1$

$$\therefore \quad 1 = \Pi_n(x) \sum_{r=0}^{n} \frac{w_{n,r}}{x - x_r}$$

or

$$\Pi_n(x) = \frac{1}{\displaystyle\sum_{r=0}^{n} \frac{w_{n,r}}{x - x_r}}$$

and we must have

$$\boxed{p_n(x) = \frac{\displaystyle\sum_{r=0}^{n} \frac{w_{n,r} f_r}{x - x_r}}{\displaystyle\sum_{r=0}^{n} \frac{w_{n,r}}{x - x_r}}}$$

Barycentric representation

where

$$w_{n,r} = [\Pi_n'(x_r)]^{-1} = \frac{1}{\displaystyle\prod_{\substack{s=0 \\ s \neq r}}^{n} (x_r - x_s)} = \frac{1}{\Pi_{n,r}(x_r)}$$

There is a special case (example sheet 1 for proof) when the interpolation points are located at the Chebyshev points

In that case
$$p_n(x) = \frac{\displaystyle\sum_{r=0}^{n}{}' \frac{\delta_r f_r}{x - x_r}}{\displaystyle\sum_{r=0}^{n}{}' \frac{\delta_r}{x - x_r}}$$

where $\delta_0 = \frac{1}{2}$, $\delta_r = (-1)^r$ $r = 1, \dots, n-1$, $\delta_n = \frac{1}{2}(-1)^n$.

Examples $[x_0, x_1, x_2] = [0, 1, 2]$

$\Pi_2 = x(x-1)(x-2)$ $\quad$ $\Pi_{2,0}(x_0) = 2$ $\quad$ $w_{2,0} = \frac{1}{2}$

$\Pi_{2,0} = (x-1)(x-2)$ $\quad$ $\Pi_{2,1}(x_1) = -1$ $\quad$ $w_{2,1} = -1$

$\Pi_{2,1} = x(x-2)$

$\Pi_{2,2} = x(x-1)$ $\quad$ $\Pi_{2,2}(x_2) = 2$ $\quad$ $w_{2,2} = \frac{1}{2}$

$$p_2(x) = \frac{\frac{\frac{1}{2}u_0}{x} - \frac{u_1}{x-1} + \frac{\frac{1}{2}u_2}{x-2}}{\frac{1}{2x} - \frac{1}{x-1} + \frac{1}{2(x-2)}}$$

If we used 3 Chebyshev pts $[-1, 0, 1]$

$\Pi_2 = (x+1)(x-1)x = \frac{1}{4}T_3$!

$\Pi_{2,0} = x(x-1)$ $\quad$ $\Pi_{2,0}(x_0) = 2$ $\quad$ $w_{2,0} = \frac{1}{2}$

$\Pi_{2,1} = (x-1)(x+1)$ $\quad$ $\Pi_{2,1}(x_1) = -1$ $\quad$ $w_{2,1} = -1$

$\Pi_{2,2} = (x+1)x$ $\quad$ $\Pi_{2,2}(x_2) = 2$ $\quad$ $w_{2,2} = \frac{1}{2}$

$$p_2(x) = \frac{\frac{\frac{1}{2}u_0}{x+1} - \frac{u_1}{x} + \frac{\frac{1}{2}u_2}{x-1}}{\frac{\frac{1}{2}}{x+1} - \frac{1}{x} + \frac{\frac{1}{2}}{x-1}}$$

6. Weierstrass Approximation Theorem (LNT ch 6)

Let $f \in C[0,1]$ then $\forall \epsilon > 0 \; \exists n, \; p_n \in P_n$ such that

$$\| f - p_n \|_\infty \le \epsilon$$

Proof: [Use Bernstein polynomials]

Define
$$b_{n,r}(x) = \frac{n!}{r!(n-r)!} x^r (1-x)^{n-r}$$

$$\beta_n(x,t) = (1 - x + xt)^n = \sum_{r=0}^{n} b_{n,r}(x) t^r$$

For $x \in [0,1]$

1. $b_{n,r} \ge 0$

2. $\left. \frac{d}{dx} b_{n,r} \right|_{\frac{r}{n}} = 0$

3. $\beta_n(x,1) = (1 - x + x)^n = 1$

   $\therefore \sum_{r=0}^{n} b_{n,r}(x) = 1 \quad \forall x \in [0,1]$ $\left\{ \begin{array}{l} \text{polynomials form} \\ \text{a partition of unity} \end{array} \right\}$

4. $\frac{\partial \beta_n}{\partial t} = nx(-x + xt)^{n-1} = \sum_{r=0}^{n} r\, b_{n,r}(x) t^r$

   $\therefore$ at $t = 1 \qquad nx = \sum_{r=0}^{n} r\, b_{n,r}(x) \quad \forall x \in [0,1]$

5. $\left. \frac{\partial^2 \beta_n}{\partial t^2} \right|_{t=1} = n(n-1)x^2 = \sum_{r=0}^{n} r(r-1) b_{n,r}(x)$

Then $\sum_{r=0}^{n}{}' (nx-r)^2 b_{n,r} = n^2x^2 \sum_{r=0}^{n}{}' b_{n,r} - 2nx \sum_{r=0}^{n}{}' r b_{n,r} + \sum_{r=0}^{n}{}' r^2 b_{n,r}$

$$= n^2x^2 - 2nx \cdot nx + \sum_{r=0}^{n}{}' r(r-1) b_{n,r} + \sum_{r=0}^{n}{}' r b_{n,r}$$

$$= -n^2x^2 + n(n-1)x^2 + nx = nx(1-x)$$

$$\therefore \boxed{\sum_{r=0}^{n}{}' \left(x - \frac{r}{n}\right)^2 b_{n,r} = \frac{x(1-x)}{n}.}$$

Next for $u \in C[0,1]$ let

$$B_n[u](x) = \sum_{r=0}^{n}{}' u\left(\frac{r}{n}\right) b_{n,r}(x) \in P_n .$$

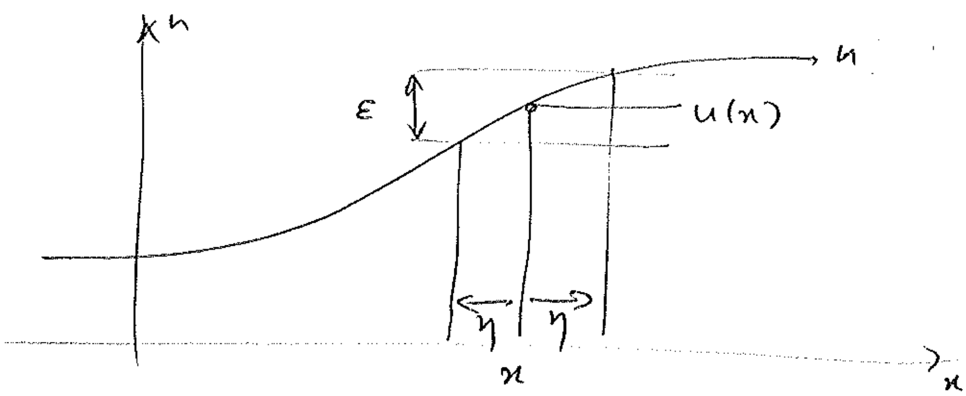
As $u \in C[0,1]$ $\|u\|_\infty$ exists and since $\sum_{r=0}^{n}{}' b_{n,r}(x) = 1$

$$u - B_n[u] = u \sum_{r=0}^{n}{}' b_{n,r} - \sum_{r=0}^{n}{}' u\left(\frac{r}{n}\right) b_{n,r}$$

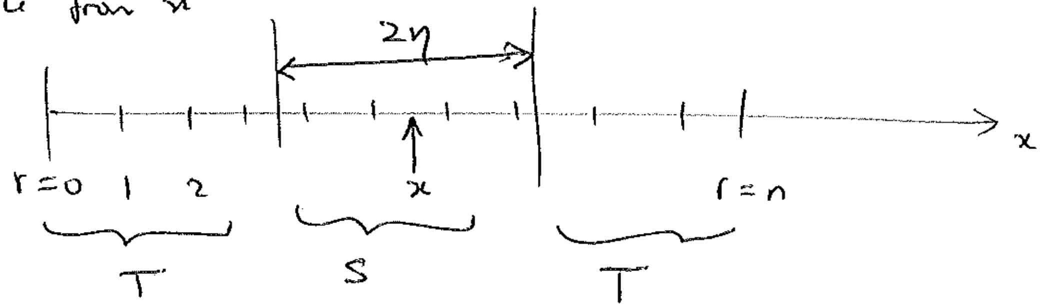
$$\therefore u - B_n[u] = \sum_{r=0}^{n}{}' b_{n,r}(x) \left[u(x) - u\left(\frac{r}{n}\right) \right]$$

As $u \in C[0,1]$ $\forall x \in [0,1]$ (x now fixed), $\forall \epsilon > 0$

$$\exists \eta > 0 \text{ st } |x - t| < \eta \Rightarrow |u(x) - u(t)| \le \frac{\epsilon}{2}$$



Next divide points $\frac{r}{n}$ $r=0,\dots,n$ according to their distance from x



Let $S = \{ r \mid |\frac{r}{n} - x| < \eta \}$

$T = \{ r \mid |\frac{r}{n} - x| \geq \eta \}$

and $\{0,\dots,n\} = S \cup T$

$$\left| \sum_{r\in S}{}' b_{n,r}(x)\left[u(x) - u\left(\tfrac{x}{r}\right)\right] \right| \leq \sum_{r\in S}{}' b_{n,r} \left| u(x) - u\left(\tfrac{r}{n}\right) \right|$$

$$\leq \frac{\varepsilon}{2} \sum_{r\in S}{}' b_{n,r} \leq \frac{\varepsilon}{2}$$

also $$\frac{x(1-x)}{n} = \sum_{r=0}^{n}{}' b_{n,r}\left(x - \tfrac{r}{n}\right)^2$$

$$\geq \sum_{r\in T}{}' b_{n,r}\left(x - \tfrac{r}{n}\right)^2 \geq \eta^2 \sum_{r\in T}{}' b_{n,r}$$

$$\therefore \sum_{r\in T}{}' b_{n,r}(x) \leq \frac{x(1-x)}{n\eta^2}$$

and $$\left| \sum_{r\in T}{}' b_{n,r}(x)\left(u(x) - u\left(\tfrac{k}{n}\right)\right) \right| \leq 2\|u\|_\infty \sum_{r\in T}{}' b_{n,r} \leq \frac{2\|u\|_\infty x(1-x)}{n\eta^2}$$

$$\therefore \left| \sum_{r\in T}{}' b_{n,r}(x)\left(u(x) - u\left(\tfrac{r}{n}\right)\right) \right| \leq \frac{\frac{1}{2}\|u\|_\infty}{n\eta^2} \quad \left[x(1-x) \leq \tfrac{1}{4} \right]$$

Choose $n > \dfrac{\|u\|_\infty}{\epsilon\eta^2}$ and

$$\|u - B_n[u]\|_\infty \leq \varepsilon.$$

Read page 43 of LNT ATAP.

③.