Existence of best approximation (See particularly Powell)

Theorem 1. If $A$ is a compact set (closed, bounded) in a metric space $B$ then

$$\forall u \in B \;\exists p \in A \text{ st } d(p,u) \le d(q,u) \;\forall q \in A$$

Proof: Let $d^* = \inf_{q \in A} d(q,u)$ and as $A$ is bounded, $d^*$ exists

difficulty is to establish that there is a $p \in A$ which achieves this bound

1). If $\exists\, p \in A$ with $d(p,u) = d^*$ then best approx exists

2) Suppose there is a sequence $p_i \in A$ with

$$\lim_{i \to \infty} d(p_i, u) = d^* \qquad (1)$$

As $A$ is compact there has to be a subsequence of $\{p_i\}$ with a limit point $p \in A$. Call the subsequence $\{\bar{p}_k\}$

$\forall \epsilon > 0$ choose $K$ such that

$k > K$ (a) $d(\bar{p}_k, u) \le d^* + \frac{1}{2}\epsilon$, this is possible because of (1)

(b) $d(\bar{p}_k, p) \le \frac{1}{2}\epsilon$, this is possible because of $p$ being a limit point

$\therefore\; d(p,u) \le d(p,\bar{p}_k) + d(\bar{p}_k, u) = d^* + \epsilon$

but $\epsilon$ arbitrary, $\therefore\; d(p,u) = d^*$

and $p \in A$ satisfies best condition,

Theorem 2: If $A$ is a finite dimensional subspace of a normed linear space $B$ then

$$\forall u \in B \ \exists p \in A \text{ st } \|u-p\| \leq \|u-q\| \ \forall q \in A$$

Proof Let $A_0 = \{ q \in A \mid \|q\| \leq 2\|u\| \}$

$A_0$ is not-empty as $0 \in A_0$

$A_0$ is compact as it is closed, bounded subset of a finite dimensional space

$\therefore \exists \ p \in A_0$ with $\|u-p\| \leq \|u-q\| \ \forall q \in A_0$

Let $q \in A$ but not $A_0$

$\|q\| = \|q-u+u\| \leq \|q-u\| + \|u\|$

$\therefore \ \|u-q\| \geq \|q\| - \|u\| > \|u\|$ as $\|q\| > 2\|u\|$

$\therefore \ \|u\| < \|u-q\|$

but $\|u-p\| \leq \|u-0\|$ as $0 \in A_0$

$\leq \|u-q\|$ if $q$ outside of $A_0$ but in $A$

$\therefore p$ is best over $A$.

## Definition convexity

A space $B$ is convex if $\forall u, v \in B$

$$w(\lambda) = (1-\lambda)u + \lambda v \in B \qquad 0 < \lambda < 1$$

## Definition Ball

Let $N(u,\epsilon) = \{ v \in B \text{ st. } \|u-v\| \le \epsilon \}$ be a closed ball of radius $\epsilon$ about $u$.

## Definition : Strict Convexity

The space $B$ is strictly convex if for $0 < \lambda < 1$, $w(\lambda)$ is an interior point (or $\forall \lambda \in (0,1)\ \exists \epsilon > 0$ st $N(w(\lambda),\epsilon) \subset B$)

**Theorem:** Let $B$ be a normed linear space, then $\forall u \in B$ then $N(u,\epsilon)$ is convex

**Proof:** $v_1, v_2 \in N \quad w = (1-\lambda)v_1 + \lambda v_2 \quad 0 < \lambda < 1$

$$\|u - w\| = \|u - (1-\lambda)v_1 - \lambda v_2\|$$
$$= \|(1-\lambda)(u-v_1) + \lambda(u-v_2)\|$$
$$\le (1-\lambda)\|u-v_1\| + \lambda\|u-v_2\| \le (1-\lambda)\epsilon + \lambda\epsilon = \epsilon$$

$\therefore w \in N(u,\epsilon)$.

The norm $\|\cdot\|$ is strictly convex if the ball defined by the norm is strictly convex.

Theorem: The 2norm of a linear inner product space $B$ is strictly convex

Proof: $u \in B$ $\quad v_1, v_2 \in N(u, \epsilon)$ $\quad v_1 \neq v_2$ and

$$\|u - v_1\|_2 = \|u - v_2\|_2 = \epsilon$$

For $0 < \lambda < 1$ let $w(\lambda) = (1-\lambda) v_1 + \lambda v_2$

Shorthand: $\quad a = u - v_1 \quad ; \|a\|_2 = \epsilon$

$\qquad\qquad\quad b = u - v_2 \quad ; \|b\|_2 = \epsilon$

and $\|a - b\|_2 = \|v_1 - v_2\|_2 \neq 0$

$$\|u - w\|_2 = \|(1-\lambda) a + \lambda b\|_2$$

$$\|(1-\lambda)a + \lambda b\|_2^2 + \lambda(1-\lambda)\|a-b\|_2^2$$

$$= \left( (1-\lambda)a + \lambda b, (1-\lambda)a + \lambda b \right) + \lambda(1-\lambda)(a-b, a-b)$$

$$= (1-\lambda)\|a\|_2^2 + \lambda \|b\|_2^2$$

$$\therefore \|u-w\|_2^2 = \|(1-\lambda)a + \lambda b\|_2^2 = (1-\lambda)\|a\|_2^2 + \lambda\|b\|_2^2 - \lambda(1-\lambda)\|a-b\|_2^2$$

$$= \epsilon^2 - \lambda(1-\lambda)\|v_1 - v_2\|_2^2 < \epsilon$$

$\therefore$ norm strictly convex.

Theorem: If $A$ is a compact strictly convex subspace of normed linear space $B$ then there is a unique best approximation from $A$ to $u \in B$.

Proof: $A$ compact $\Rightarrow$ best approximation exists

Assume two distinct best approximations

Let $p_1, p_2 \in A$ with $p_1 \neq p_2$ and

$$\|u-p_1\| = \|u-p_2\| = d^* = \inf_{q\in A} \|u-q\|$$

Let $w = \frac{1}{2}(p_1+p_2)$

$$\|u-w\| = \|u - \tfrac{1}{2}p_1 - \tfrac{1}{2}p_2\| = \|\tfrac{1}{2}(u-p_1) + \tfrac{1}{2}(u-p_2)\|$$
$$\leq \tfrac{1}{2}\|u-p_1\| + \tfrac{1}{2}\|u-p_2\|$$
$$\leq d^*,$$

but $\|u-w\|$ cannot be less than $d^*$

$\therefore \|u-w\| = d^*$ and $w$ also best

A is strictly convex so $w$ has to be an interior point of A

$\therefore \exists\, \delta > 0$ st $N(w, \delta\|u-w\|) \subset A$ (implicitly $\delta \leq 1$)

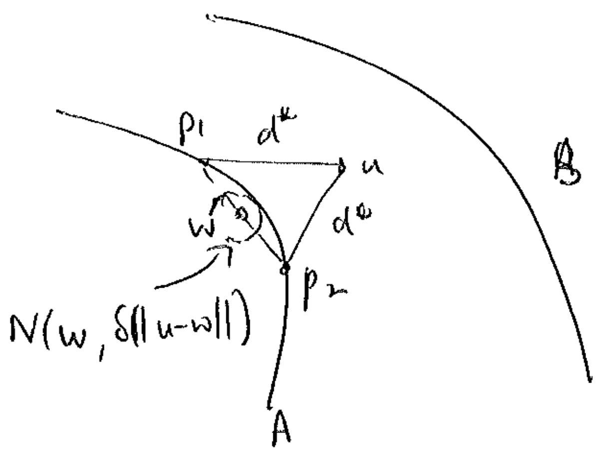
Let $s = w + \delta(u-w)$ which is in N and therefore A

$$\|u-s\| = \|u-w-\delta(u-w)\| \leq (1-\delta)\|u-w\|$$

$\therefore \|u-s\| \leq (1-\delta)d^*$ and $s \in A$

but this contradicts $d^* = \inf_{q\in A}\|u-q\|$

$\therefore$ best approximation is unique.



Definition Let $A$ be a compact set in a normed linear space $B$

Let $X : B \to A$ be a linear operator with

$\forall u \in B \quad q = X[u] \in A$ and a projection

property $\forall p \in A \quad X[p] = p$

Define $$\|X\| = \sup_{u \in B} \frac{\|X[u]\|}{\|u\|}$$

$\|X\|$ is called the Lebesgue constant of operator $X$.

$\|X\|$ may not exist.

Theorem: Let $d^* = \inf_{q \in A} \|u - q\|$ for $u \in B$, then

$$d^* \leq \|u - X[u]\| \leq (1 + \|X\|) d^*$$

Proof 1) $\|u - X[u]\| \geq d^*$ by definition of $d^*$

2) $\|u - X[u]\| = \|u - p + p - X[u]\|$ (p is best)

$= \|u - p + X[p] - X[u]\|$ (X is projection operator)

$= \|u - p - X[u-p]\|$ (X is linear)

$\leq \|u - p\| + \|X[u-p]\|$

$\leq \|u - p\| + \|X\| \, \|u - p\|$

$\therefore \|u - X[u]\| \leq (1 + \|X\|) d^*$

Example! $Z_n$ = interpolation at $n+1$ uniformly spaced points

$Y_n$ = interpolation at roots of $T_{n+1}$

| $n$ | $\|Z_n\|$ | $\|Y_n\|$ |
|---|---|---|
| 2 | 1.25 | 1.25 |
| 10 | 29.9 | 2.07 |
| 20 | 10986 | 2.48 |

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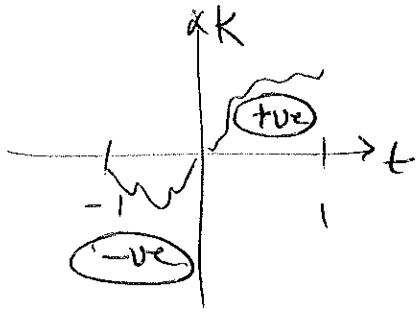
## Calculation of Lebesgue constants

Suppose $$X[u] = \int_{-1}^{1} u(t)\,K(t)\,dt$$

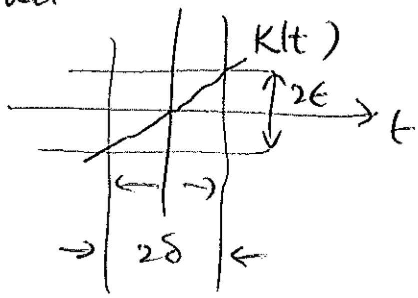
where $K(t)\begin{cases} \geq 0 & x \geq 0 \\ < 0 & x < 0 \end{cases}$

and $K$ continuous.




As $K$ continuous, $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$|t| \leq \delta \Rightarrow |K| \leq \epsilon$$

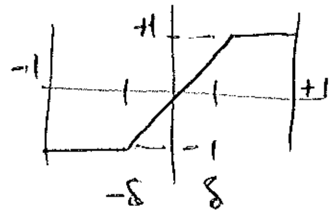



(a) $$|X[u]| = \left| \int_{-1}^{1} K(t)u(t)\,dt \right|$$
$$\leq \|u\|_\infty \int_{-1}^{1} |K(t)|\,dt$$

$$\therefore \frac{|X[u]|}{\|u\|_\infty} \leq \int_{-1}^{1} |K(t)|\,dt \quad \forall u \in C[-1,1] \text{ with } \|u\| > 0$$

$$\therefore \|X\|_\infty \leq \int_{-1}^{1} |K(t)|\,dt$$

(b) Choose $u(t) = \begin{cases} -1 & -1 < t < \delta \\ t/\delta & -\delta \le t \le \delta \\ +1 & \delta < t < 1 \end{cases}$




so that $u(t) \in C[-1,1]$

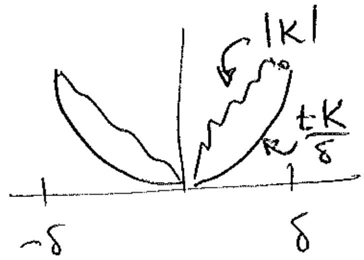
$$X[u] = \int_{-1}^{1} u(t)K(t)\,dt = \int_{-1}^{-\delta} |K(t)|\,dt + \int_{\delta}^{1} |K(t)|\,dt$$

$$+ \int_{-\delta}^{\delta} \frac{tK}{\delta}\,dt$$

$$\therefore X[u] = \int_{-1}^{1} |K(t)|\,dt - \int_{-\delta}^{\delta} \left(|K| - \frac{t}{\delta}K\right) dt$$

but in $-\delta \le t \le \delta$ $|K| \le \epsilon$ and $1 - \frac{t}{\delta} \le 1$

$$\therefore \int_{-\delta}^{\delta} \left(|K| - \frac{tK}{\delta}\right) dt \le 2\epsilon\delta$$




$$\therefore X[u] \ge \int_{-1}^{1} |K(t)|\,dt - 2\epsilon\delta$$

and $\|u\|_\infty = 1$ for this example function

Since $\|X\|_\infty = \sup_u \frac{\|X[u]\|_\infty}{\|u\|_\infty}$

$$\|X\|_\infty \ge \frac{\|X[u]\|_\infty}{\|u\|_\infty} \quad \text{for any particular } u$$

$$\therefore \int_{-1}^{1} |K(t)|\,dt - 2\epsilon\delta \le \|X\|_\infty \le \int_{-1}^{1} |K(t)|\,dt$$

but $\epsilon, \delta$ arbitrarily small $\therefore \|X\|_\infty = \int_{-1}^{1} |K(t)|\,dt$.