Equioscillation Theorem (ATAP ch10)

Also sometimes called Oscillation Theorem or Characterisation Theorem

Assume that a best approximation in L_∞ norm exists, denote $p_n \in P_n$ and let $E_n = \|u - p_n\|_\infty$.

Theorem: Let $u \in C[-1,1]$ then $p_n \in P_n$ is the unique best L_∞ approximation from P_n to u iff there is a sequence of $n+2$ distinct ordered points $x_0 \ldots x_{n+1}$ in $[-1,1]$ such that

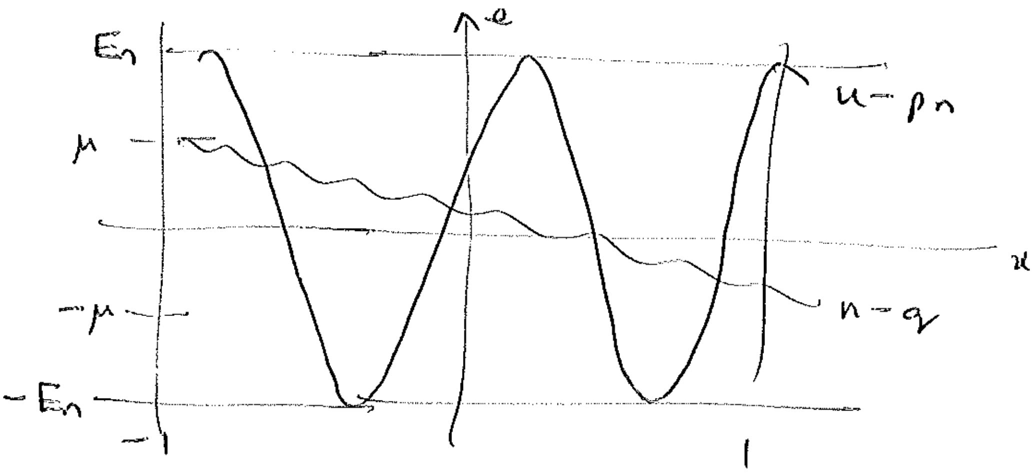
$$u(x_r) - p_n(x_r) = \pm E_n \qquad r = 0, \ldots, n+1$$

and alternating in sign.

Proof: Let $e = u - p_n$ be the error so that $\|e\|_\infty = E_n$

(1) Assume $p_n \in P_n$ has this oscillation property.

Assume $\exists q \in P_n$ with $\|u - q\|_\infty \le \mu < E_n$



Then $p_n - q = (u - q) - (u - p_n)$ and at extrema of $u - p_n$

$$u - p_n = E_n \quad p_n - q = (u - q) - E_n < 0$$

$$u - p_n = -E_n \quad p_n - q = (u - q) + E_n > 0$$

∴ $p_n - q$ has $n+2$ alternating values

but $p_n - q \in P_n$ so continuous

∴ $p_n - q$ has $n+1$ roots in $[-1,1]$

which contradicts $p_n - q \in P_n$

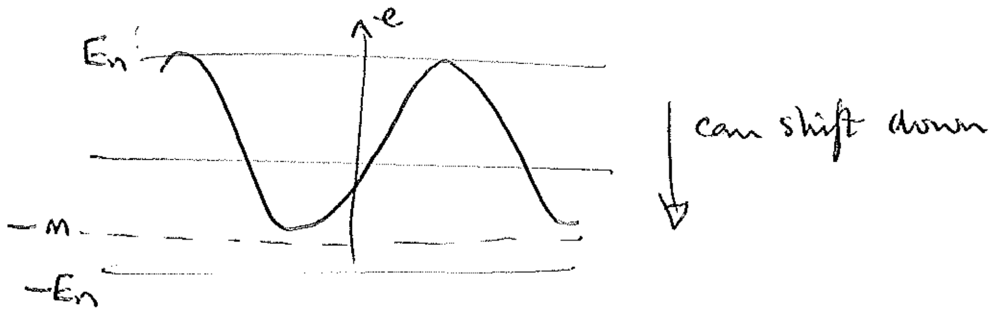
∴ no such q exists

(2) Suppose $p_n \in P$ is best; want to show it must have oscillation property

Define $N = \{x \mid u(x) - p_n(x) = E_n\}$

$M = \{x \mid u(x) - p_n(x) = -E_n\}$

(a) Assume M empty with $m > 0$, $-m = \inf(u - p_n) > -E_n$



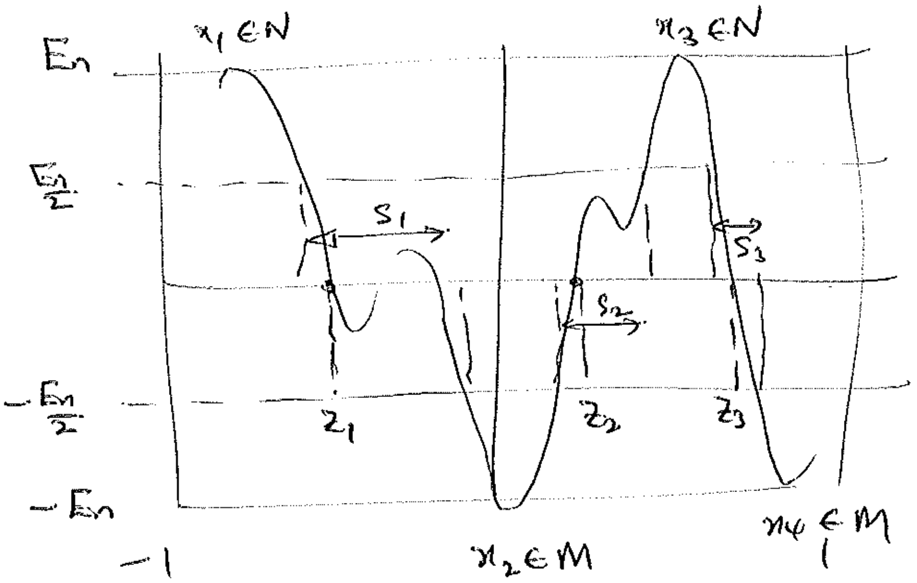
then $\|u - p_n - \frac{E_n - m}{2}\|_\infty = \frac{E_n + m}{2} < E_n$ which contradicts p_n being best

∴ M is not empty, similarly N is not empty

Let the points in M, N be ordered $x_1, \dots, x_{k+1}$

As u, p_n continuous, $\exists$ points $z_1 \dots z_k$ between points in M, N such that $e(z_r) = 0$ $r = 1, \dots, k$.

(choose only one zero between each pair of points)



For each point z_r with $x_r < z_r < x_{r+1}$

$$S_r = \left\{ x \;\middle|\; x_r < x < x_{r+1} \text{ and } |e(x)| \le \frac{E_n}{2} \right\}$$

$$S = \bigcup_{r=1}^{k} S_r \quad \text{with} \quad \sigma = \sup_{x \in S} |e| = \frac{E_n}{2}$$

For $x \notin S$ $\quad e > \frac{E_n}{2}$ near points in N

$e < -\frac{E_n}{2}$ near points in M.

Define $\quad \phi(x) = \pm \prod_{r=1}^{k} (x - z_r)$ so that sign ϕ opposite to sign e at points in M, N

Let $\quad \rho = \|\phi\|_\infty \qquad \lambda = \frac{E_n}{4\rho}$

Define $\quad q(x) = p_n(x) - \lambda \phi(x)$

(a) $x \in S$ $\quad |u-q| = |u - p_n + \lambda \phi|$

$$\leq |u - p_n| + \lambda |\phi|$$

$$\leq \frac{E_n}{2} + \frac{E_n}{4\rho}\rho = \frac{3}{4}E_n < E_n$$

(b) $x \notin S$, cannot be near roots of ϕ so that

$$\exists \mu > 0 \text{ st } \quad \mu\rho \leq |\phi| \leq \rho$$

(i) $\phi < 0$ $\qquad \frac{E_n}{2} < u - p_n \leq E_n$

$$-\lambda\rho \leq \lambda\phi \leq -\lambda\mu\rho$$

$$\frac{E_n}{2} - \lambda\rho \leq u - p_n + \lambda\phi \leq E_n - \lambda\mu\rho$$

$$\frac{E_n}{4} \leq u - q \leq E\left(1 - \frac{\mu}{4}\right) < E \text{ while } \mu > 0$$

(ii) $\phi > 0$ $\qquad -E_n \leq u - p_n < -\frac{E_n}{2}$

$$\lambda\mu\rho \leq \lambda\phi \leq \lambda\rho$$

$$-E_n < -\left(1 - \frac{\mu}{4}\right)E_n \leq u - q \leq -\frac{E_n}{4}$$

$$\therefore x \notin S \quad |u - q| < E_n$$

This would contradict p_n being best unless q is not a P_n, ie $q \in P_{n+m}$ where $m \geq 1$

$\therefore$ $k \geq n+2$ and error has at least $n+2$ alternating extrema.

(3) uniqueness

Let p_n, q_n be distinct with $E_n = \|u - p_n\| = \|u - q_n\|$

Let $h = \frac{1}{2}(p_n + q_n)$

$u - h = \frac{1}{2}(u - p_n) + \frac{1}{2}(u - q_n)$

$\therefore \|u - h\|_\infty \le \frac{1}{2}E_n + \frac{1}{2}E_n = E_n$

but $\|u - h\|_\infty$ cannot be less than E_n

$\therefore \|u - h\|_\infty = E_n$ and h also best

$\therefore h$ has $n+2$ alternating extreme where $u - h = \pm E_n$

but when $u - h = E_n$

$\frac{1}{2}(u - p_n) + \frac{1}{2}(u - q_n) = E_n$

and since $\frac{1}{2}(u - p_n) \le E_n$, $\frac{1}{2}(u - q_n) \le E_n$

have to also have $u - p_n = E_n$, $u - q_n = E_n$

Similarly if $u - h = -E_n$ so too must $u - p_n$, $u - q_n$

$\therefore p_n - q_n = (u - q_n) - (u - p_n) = 0$ at $n+2$ points

but $p_n - q_n \in P_n$ so this is not possible

and p_n must be unique.

Theorem: de la Vallee Poussin

Let $p_n \in P_n$ be the best L_∞ approximation to $u \in C[-1,1]$

write $E_n = \|u - p_n\|_\infty$. Let $q \in P_n$ with $M = \|u - q\|_\infty$

have the property that $u - q$ has a sequence of

$n+2$ alternating extrema at distinct points $x_0 \dots x_{n+1}$ in $[-1,1]$,

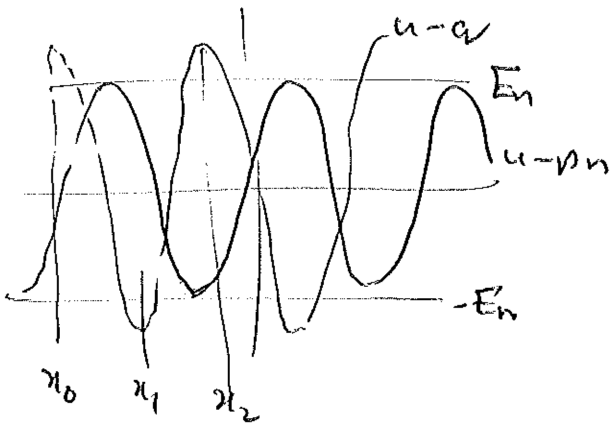
Assume that at one point in this set the error equals M.

Then

$$m = \min_r |u(x_r) - q(x_r)| \leq E_n \leq M = \max_r |u(x_r) - q(x_r)|$$

Proof (1) $M \geq E_n$ as p_n is best

(2) Suppose $m > E_n$



At any point x_r

$p_n(x_r) - q(x_r)$

$= (u - q(x_r)) - (u(x_r) - p_n(x_r))$

(a) if $u - q > 0$ $\quad p_n - q > 0$

(b) if $u - q < 0$ $\quad p_n - q < 0$

$\therefore$ $p_n - q$ must have $n+1$ roots which contradicts

$p_n - q \in P_n$

$\therefore$ $m \leq E_n$.